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Intervened Generalized Gegenbauer Distribution and its Properties

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ABSTRACT

In this paper, an intervened version of the generalized Gegenbauer distribution is considered and studied some of its statistical properties. The parameters of the distribution are estimated by using the method of maximum likelihood, method of mixed moments and illustrated using real life data sets. The likelihood ratio test procedure is applied for examining the significance of the intervention parameters and a simulation study is carried out for assessing the performance of the estimators.

KEYWORDS
Generalized Gegenbauer distribution, generalized Gegenbauer polynomials, intervened negative binomial distribution, maximum likelihood estimation, probability generating function.

1. Introduction

Many distributions in statistical literature arise as mixture of other distributions. [4] obtained the generalized Gegenbauer distribution (GGbD) by mixing generalized Hermite distribution with gamma distribution; they refer to it as generalized Gegenbauer distribution (GGbD) because of its relationship with generalized Gegenbauer polynomial. The generalized Gegenbauer polynomial defined through the generating function
$(1-\alpha s-\beta s^m)^{-\lambda}=\sum_{n=0}^{\infty}\omega_{n,m}^{\lambda}(\alpha,\beta)s^m$ for$\lambda>0$, $\alpha>0$ and $\beta>0$ is

$$\omega_{n,m}^{\lambda}(\alpha,\beta)=\sum_{j=0}^{[\frac{n}{m}]}\frac{\Gamma(\lambda+n-mj+j)}{\Gamma(n-mj+j+1)\Gamma(\lambda)\Gamma(j+1)}\alpha^{n-mj}\beta^{j} \tag{1}$$

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The probability generating function of GGbD proposed by [4] is

$$G(s) = (1-\theta_1-\theta_2)^r(1-\theta_1 s-\theta_2 s^m)^{-r} \tag{2}$$

in which $\theta_1 > 0$, $\theta_2 > 0$, $r > 0$, $\theta_1+\theta_2 < 1$ and m is a positive integer.

When m=1 GGbD reduces to negative binomial distribution and when m=2 it becomes the distributions of [5], [7]. Through this paper we propose an intervened version of the generalized Gegenbauer distribution and termed it as "the intervened generalized Gegenbauer distribution" or in short "the IGGbD". It is shown that the IGGbD includes the IGD of [1] , the ZTNBD and the INBD of [3] as its special cases. The paper is organised in such a way that in section 2 we develop a model leading to the IGGbD and derive expressions for the pmf, moments, mean, variance and recurrence relations of probabilities . In section 3 we discuss the estimation of the parameters of the IGGbD. In section 4, likelihood ratio test procedure is used for testing the significance of the intervention parameters. A simulation study is carried out in section 5 for examining the performance of the estimators of the parameters of the IGGbD.
We need the following in the sequel. For any real valued function $A(i,r)$, we have

$$\sum_{i=0}^{\infty}\sum_{r=0}^{\infty} A(i,r) = \sum_{i=0}^{\infty}\sum_{r=0}^{i} A(i-r,r) \tag{3}$$

2. Intervened Generalized Gegenbauer Distribution

Let Z be a random variable having zero truncated generalized Gegenbauer distribution with parameters r, θ_1 and θ_2. Then the pgf of Z is

$$P_Z(s) = \frac{(1-\theta_1-\theta_2)^r}{1-(1-\theta_1-\theta_2)^r}[(1-\theta_1 s-\theta_2 s^m)^{-r}-1], \tag{4}$$

in which $r > 0$, and $\theta_i > 0$ for $i = 1, 2$ such that $\theta_1+\theta_2 \leq 1$ and m is a positive integer. Let Y be a random variable following generalized Gegenbauer distribution in which, due to some interventions, the parameters θ_1 changes to $\rho_1\theta_1$ and θ_2 changes to $\rho_2\theta_2$ with $\rho_1 > 0$, $\rho_2 > 0$ such that $\rho_1\theta_1+\rho_2\theta_2 \leq 1$ and the parameters ρ_1 and ρ_2 are called the intervention parameters. Assume that Y and Z are statistically independent. Then the pgf of $X = Y + Z$ is given by
$P_X(s) = P_Y(s)P_Z(s)$

$$= \frac{(1-\rho_1\theta_1-\rho_2\theta_2)^r}{(1-\theta_1-\theta_2)^{-r}-1}[(1-\theta_1 s-\theta_2 s^m)^{-r}-1][1-\rho_1\theta_1 s-\rho_2\theta_2 s^m]^{-r}, \tag{5}$$

in which $r > 0$, $\rho_i > 0$ and $\theta_i > 0$ for $i = 1, 2$ such that $\rho_1\theta_1 + \rho_2\theta_2 \leq 1$. The distribution of a random variable X with pgf (5), we call "the intervened generalized Gegenbauer distribution" or in short "the IGGbD". Clearly, when $m = 1$,$\theta_1+\theta_2 = \theta$ and $\rho_1, \rho_2 = \rho$ the pgf (5) reduces to the pgf of the INBD of [3].
Now we obtain the pmf of the IGGbD through the following proposition.

Proposition 2.1. *Let X follows IGGbD with pgf as given in (5), then the pmf g_x of X, for $x = 1, 2, 3, ...$ is*

$$g_x = \frac{(1-\rho_1\theta_1-\rho_2\theta_2)^r}{(1-\theta_1-\theta_2)^{-r}-1} \sum_{i=0}^{x-1} \omega^r_{x-i,m}(\theta_1,\theta_2)\ \omega^r_{i,m}(\rho_1\theta_1,\rho_2\theta_2) \tag{6}$$

in which $r > 0$, $\rho_i > 0$, $\theta_i > 0$ for $i = 1, 2$ such that $\rho_1\theta_1 + \rho_2\theta_2 \leq 1$and $\omega^\lambda_{n,m}(\alpha, \beta)$ is the generalized Gegenbauer polynomial as defined in (1) and m is a positive integer.

Proof. By definition, the pgf $P_X(s)$ of the IGGbD is

$$P_X(s) = \sum_{x=0}^{\infty} s^x g_x, \tag{7}$$

$$= \frac{(1-\rho_1\theta_1-\rho_2\theta_2)^r}{(1-\theta_1-\theta_2)^{-r}-1}[(1-\theta_1 s-\theta_2 s^m)^{-r}-1][1-\rho_1\theta_1 s-\rho_2\theta_2 s^m]^{-r}. \tag{8}$$

Applying (1) in (8), we obtain the following.

$$P_X(s) = \frac{(1-\rho_1\theta_1-\rho_2\theta_2)^r}{(1-\theta_1-\theta_2)^{-r}-1}\sum_{x=0}^{\infty}\omega^r_{x,m}(\theta_1,\theta_2)s^x\sum_{i=0}^{\infty}\omega^r_{i,m}(\rho_1\theta_1,\rho_2\theta_2)s^i - \frac{(1-\rho_1\theta_1-\rho_2\theta_2)^r}{(1-\theta_1-\theta_2)^{-r}-1}\sum_{x=0}^{\infty}\omega^r_{x,m}(\rho_1\theta_1,\rho_2\theta_2)s^x \tag{9}$$

$$= \frac{(1-\rho_1\theta_1-\rho_2\theta_2)^r}{(1-\theta_1-\theta_2)^{-r}-1}\sum_{x=0}^{\infty}\sum_{i=0}^{x}\omega^r_{x-i,m}(\theta_1,\theta_2)\omega^r_{i,m}(\rho_1\theta_1,\rho_2\theta_2)s^x - \frac{(1-\rho_1\theta_1-\rho_2\theta_2)^r}{(1-\theta_1-\theta_2)^{-r}-1}\sum_{x=0}^{\infty}\omega^r_{x,m}(\rho_1\theta_1,\rho_2\theta_2)s^x \tag{10}$$

in the light of (3). Now, on equating the coefficients of s^x in the right hand side expressions of (7) and (10), we get (6). □

Next, we develop an expression for the p^{th} factorial moment of the $IGGbD(r,\ \rho_1,\ \rho_2,\ \theta_1,\ \theta_2)$ through the following result.

Result 2.1. *The p^{th} factorial moment $\mu'_{(p)}$ of the $IGGbD(r,\ \rho_1,\ \rho_2,\ \theta_1,\ \theta_2)$ with pgf given in (6) is*

$$\mu'_{(p)} = \frac{p!\eta^*}{[\Gamma(r)]^2}\sum_{J}\sum_{I_{j_2}}\frac{\Gamma(j-j_1+j_2-j_3+r)\Gamma(j_1-j_2+j_3+r)(\Gamma j+1)}{\Gamma(j_2-j_3+1)\Gamma(j_3+1)\Gamma(j-j_1+1)\Gamma(j_1-j_2+1)}\delta^j_{11}\left(\frac{\delta_{21}}{\delta_{11}}\right)^{j_1} \left(\frac{\delta_{12}}{\delta_{21}}\right)^{j_2}\left(\frac{\delta_{22}}{\delta_{12}}\right)^{j_3}\left(\prod_{i=1}^{m}\frac{\binom{m}{i}^{a_i}}{a_i!}\right) + \frac{(1-\eta^*)}{\Gamma(r)}\sum_{J*}\sum_{I_{j_2}}\frac{\Gamma(j+r)\Gamma(j_2+1)}{\Gamma(j+1)} \delta^j_{21}\left(\frac{\delta_{22}}{\delta_{21}})\right)^{j_2}\left(\prod_{i=1}^{m}\frac{\binom{m}{i}^{a_i}}{a_i!}\right), \tag{11}$$

in which, $\eta^* = [1-(1-\theta_1-\theta_2)^r]^{-1}$, $\delta_{11} = \frac{\theta_1}{1-\theta_1-\theta_2}$, $\delta_{12} = \frac{\theta_2}{1-\theta_1-\theta_2}$, $\delta_{21} = \frac{\rho_1\theta_1}{1-\rho_1\theta_1-\rho_2\theta_2}$, $\delta_{22} = \frac{\rho_2\theta_2}{1-\rho_1\theta_1-\rho_2\theta_2}$,

$$J = \{(j, j_1, j_2, j_3) \in N^4 : 0 \le j < \infty, 0 \le j_1 < j, 0 \le j_2 < j_1, \\ 0 \le j_3 < j_2,\ \tau + j - j_2 = p\}$$

$J* = \left\{(j, j_2) \in N^2 : 0 \le j < \infty,\ 0 \le j_2 < j,\ \ \tau + j - j_2 = p\right\}$ *and* $\sum_{I_{j_2}}$ *denotes the summation over* m *non- negative integers in the set*
$J_2 = \left\{(a_1, a_2, ...a_m) : \sum_{i=1}^{m} a_i = j_2, \tau = \sum_{i=1}^{m} i a_i\right\}$.

Proof. The factorial moment generating function $H_X(t)$ of the random variable X is

$$H_X(t) = \sum_{p=0}^{\infty} \mu'_{(p)} \frac{t^p}{p!}. \tag{12}$$

By replacing s by $1+t$ in (8), we get the $H_X(t)$ of $IGGbD(r,\ \rho_1,\ \rho_2,\ \theta_1,\ \theta_2)$ as

$\Lambda_2\ (r,\ \rho,\ \theta)\,[(1-\theta_1(1+t)-\theta_2(1+t)^m)^{-r}-1]\,[1-\rho_1\theta_1(1+t)-\rho_2\theta_2(1+t)^m]^{-r}$,
where $\Lambda_2\ (r,\ \rho,\ \theta) = \frac{(1-\rho_1\theta_1-\rho_2\theta_2)^r}{(1-\theta_1-\theta_2)^{-r}-1}$.

On rearranging and substituting η^*, δ_{11}, δ_{12}, δ_{21} and δ_{22}, we get

$$\begin{aligned} H_X(t) &= \eta^* \left(1-\delta_{11}t-\delta_{12}\sum_{i=1}^{m}\binom{m}{i}t^i\right)^{-r}\left(1-\delta_{21}t-\delta_{22}\sum_{i=1}^{m}\binom{m}{i}t^i\right)^{-r} \\ &\quad +(1-\eta^*)\left(1-\delta_{21}t-\delta_{22}\sum_{i=1}^{m}\binom{m}{i}t^i\right)^{-r}. \end{aligned} \tag{13}$$

By using the negative binomial expansion in (13), we obtain the following.

$$H_X(t) = \eta^* \sum_{j=0}^{\infty}\sum_{j_1=0}^{\infty}\binom{j+r-1}{j}\binom{j_1+r-1}{j_1}\left(\delta_{11}t+\delta_{12}\sum_{i=1}^{m}\binom{m}{i}t^i\right)^{j} \\ \left(\delta_{21}t+\delta_{22}\sum_{i=1}^{m}\binom{m}{i}t^i\right)^{j_1} + (1-\eta^*)\sum_{j=0}^{\infty}\left(\delta_{21}t+\delta_{22}\sum_{i=1}^{m}\binom{m}{i}t^i\right)^{j}.$$

$$H_X(t) = \eta^* \sum_J \binom{j-j_1+j_2-j_3+r-1}{j-j_1+j_2-j_3} \binom{j_1-j_2+j_3+r-1}{j_1-j_2+j_3} \binom{j-j_1+j_2-j_3}{j_2-j_3}$$
$$\binom{j_1-j_2+j_3}{j_3} \delta_{11}^j \left(\frac{\delta_{21}}{\delta_{11}}\right)^{j_1} \left(\frac{\delta_{12}}{\delta_{21}}\right)^{j_2} \left(\frac{\delta_{22}}{\delta_{21}}\right)^{j_3} t^{j-j_2} \left(\sum_{i=1}^m \binom{m}{i} t^i\right)^{j_2}$$
$$+(1-\eta^*) \sum_{j=0}^{\infty} \sum_{j_2=0}^{j} \binom{j+r-1}{j} \binom{j}{j_2} \delta_{21}^j \left(\frac{\delta_{22}}{\delta_{21}})\right)^{j_2} t^{j-j_2} \left(\sum_{i=1}^m \binom{m}{i} t^i\right)^{j_2},$$

where on $\sum_J$
the set $J = \{j, j_1, j_2, j_3 \in N^4 : 0 < j < \infty, 0 < j_1 < j, 0 < j_2 < j_1, 0 < j_3 < j_2\}$.
Again rearranging, we get

$$\begin{aligned}
H_X(t) &= \frac{\eta^*}{\Gamma(r)^2} \sum_J \frac{\Gamma(j-j_1+j_2-j_3+r)\ \Gamma(j_1-j_2+j_3+r)}{\Gamma(j_2-j_3+1)\ \Gamma(j_3+1)\ \Gamma(j-j_1+1)\ \Gamma(j_1-j_2+1)} \\
&\quad (\delta_{11})^j \left(\frac{\delta_{21}}{\delta_{11}}\right)^{j_1} \left(\frac{\delta_{12}}{\delta_{21}}\right)^{j_2} \left(\frac{\delta_{22}}{\delta_{12}}\right)^{j_3} t^{j-j_2} \left(\sum_{i=1}^m \binom{m}{i} t^i\right)^{j_2} \\
&\quad + \frac{(1-\eta^*)}{\Gamma(r)} \sum_{j=0}^{\infty} \sum_{j_2=0}^{j} \frac{\Gamma(j+r)}{\Gamma(j+1)} (\delta_{21})^j \left(\frac{\delta_{22}}{\delta_{21}})\right)^{j_2} (t)^{j-j_2} \left(\sum_{i=1}^m \binom{m}{i} t^i\right)^{j_2} \\
&= \frac{\eta^*}{[\Gamma(r)]^2} \sum_J \sum_{I_{j_2}} \frac{\Gamma(j-j_1+j_2-j_3+r)\Gamma(j_1-j_2+j_3+r)(\Gamma j+1)}{\Gamma(j_2-j_3+1)\Gamma(j_3+1)\Gamma(j-j_1+1)\Gamma(j_1-j_2+1)} \\
&\quad \delta_{11}^j \left(\frac{\delta_{21}}{\delta_{11}}\right)^{j_1} \left(\frac{\delta_{12}}{\delta_{21}}\right)^{j_2} \left(\frac{\delta_{22}}{\delta_{12}}\right)^{j_3} \left(\prod_{i=1}^m \frac{\binom{m}{i}^{a_i}}{a_i!}\right) t^{\tau+j-j_2} + \frac{(1-\eta^*)}{\Gamma(r)} \\
&\quad \sum_{J*} \sum_{I_{j_2}} \frac{\Gamma(j+r)\Gamma(j_2+1)}{\Gamma(j+1)} \delta_{21}^j \left(\frac{\delta_{22}}{\delta_{21}})\right)^{j_2} \left(\prod_{i=1}^m \frac{\binom{m}{i}^{a_i}}{a_i!}\right) t^{\tau+j-j_2}, \qquad (14)
\end{aligned}$$

in the light of multinomial theorem, where
$J* = \{(j, j_2) \in N^2 : 0 \le j < \infty,\ 0 \le j_2 < j,\ \ \tau + j - j_2 = p\}$ and $\sum_{I_{j_2}}$ denotes the summation over m non- negative integers in the set
$J_2 = \left\{(a_1, a_2, ...a_m) : \sum_{i=1}^m a_i = j_2, \tau = \sum_{i=1}^m i a_i\right\}$. Now, on equating the coefficients of $\frac{t^p}{p!}$ in the right hand side expressions of (12) and (14), we get (11). □

Result 2.2. *The Mean and Variance of the IGGbD*$(r,\ \rho_1,\ \rho_2,\ \theta_1,\ \theta_2)$ *are*

$$\begin{aligned} Mean &= r\delta_{21} + mr\delta_{22} + \eta^*(r\delta_{11} + mr\delta_{12}) \quad (15) \\ and & \\ Variance &= r\delta_{21}(1+\delta_{21}) + m^2 r\delta_{22}(1+\delta_{22}) + r\eta^*\delta_{11}(1+\delta_{11}) + m^2 r\eta^*\delta_{12}(1+\delta_{12}) \\ &\quad + r^2\eta^*(1-\eta^*)(\delta_{11}+m\delta_{12})^2 + 2mr(\delta_{21}\delta_{22} + \eta^*\delta_{11}\delta_{12}), \quad (16) \end{aligned}$$

in which η^*, δ_{11}, δ_{12}, δ_{21} *and* δ_{22} *are as defined in (11).*

Result 2.3. *If* $r\eta^* < 1$ *then IGGbD*$(r,\ \rho_1,\ \rho_2,\ \theta_1,\ \theta_2)$ *is over-dispersed and if* $r\eta^* > 1$ *then it is over dispersed if*

$$\begin{aligned} (\delta_{11}^2 + m^2\delta_{12}^2 + 2m\delta_{11}\delta_{12})(r\eta^*(1-r\eta^*)) &< r\delta_{21}^2 + rm^2\delta_{22}^2 + m(m-1)(r\delta_{22} + r\eta^*\delta_{12}) \\ &\quad + 2mr(\delta_{21}\delta_{22} + r\eta^*\delta_{11}\delta_{12}) + r^2\eta^*\delta_{11}^2 + m^2r^2\eta^*\delta_{12}^2 \end{aligned}$$

and under dispersed if

$$\begin{aligned} (\delta_{11}^2 + m^2\delta_{12}^2 + 2m\delta_{11}\delta_{12})(r\eta^*(1-r\eta^*)) &> r\delta_{21}^2 + rm^2\delta_{22}^2 + m(m-1)(r\delta_{22} + r\eta^*\delta_{12}) \\ &\quad + 2mr(\delta_{21}\delta_{22} + r\eta^*\delta_{11}\delta_{12}) + r^2\eta^*\delta_{11}^2 + m^2r^2\eta^*\delta_{12}^2 \end{aligned}$$

Result 2.4. *The following is a recurrence relation for probabilities of the IGGbD*$(r,\ \rho_1,\ \rho_2,\ \theta_1,\ \theta_2)$, *for* $x \geq 1$.

$$\begin{aligned} x\ g(x+1) &= r\rho_1\theta_1 \sum_{i=0}^{x} \omega_{i,\ m}^{1}\ (\rho_1\theta_1,\ \rho_2\theta_2)\ g(x-i) \\ &\quad + mr\rho_2\theta_2 \sum_{i=0}^{x-m+1} \omega_{i,\ m}^{1}\ (\rho_1\theta_1,\ \rho_2\theta_2)\ g(x-i-m+1) \\ &\quad + B(x: r, m, \rho_1, \rho_2, \theta_1, \theta_2) \quad (17) \end{aligned}$$

in which,

$$\begin{aligned} B(x: r, m, \rho_1, \rho_2, \theta_1, \theta_2) &= \Lambda_2\ (r,\ \rho,\ \theta) \left[r\theta_1 \sum_{i=0}^{x} \omega_{x-i,\ m}^{r}\ (\rho_1\theta_1,\ \rho_2\theta_2)\ \omega_{i,\ m}^{r+1}\ (\theta_1,\ \theta_2) \right. \\ &\quad \left. + mr\theta_2 \sum_{i=0}^{x-m+1} \omega_{x-i-m+1,\ m}^{r}\ (\rho_1\theta_1,\ \rho_2\theta_2)\ \omega_{i,\ m}^{r+1}\ (\theta_1,\ \theta_2) \right] \end{aligned}$$

and $\omega_{n,\ m}^{r}\ (a,\ b)$ *is the generalized Gegenbauer polynomial as defined in (1).*

Proof. From (8), we have

$$P_X(s) = \sum_{x=0}^{\infty} s^x g(x) \quad (18)$$

$$= \Lambda_2\ (r,\ \rho,\ \theta) \left[(1-\theta_1 s - \theta_2 s^m)^{-r} - 1 \right] \left[1 - \rho_1\theta_1 s - \rho_2\theta_2 s^m \right]^{-r}. \quad (19)$$

On differentiating (18) and (19) with respect to s, we get the following.

$$\sum_{x=0}^{\infty}(x+1)s^x g(x+1) = \frac{r\left(\rho_1\theta_1 + m\rho_2\theta_2 s^{m-1}\right)}{1-\rho_1\theta_1 s-\rho_2\theta_2 s^m} p_X(s)$$
$$+\Lambda_2\ (r,\ \rho,\ \theta)\frac{r\left(\theta_1 + m\theta_2 s^{m-1}\right)}{(1-\theta_1 s-\theta_2 s^m)^{-r-1}(1-\rho_1\theta_1 s-\rho_2\theta_2 s^m)^{-r}}$$

Now, on rearranging, we get

$$\sum_{x=0}^{\infty}(x+1)s^x\ g(x+1) = r\rho_1\theta_1\sum_{x=0}^{\infty}\sum_{i=0}^{x}\omega_{i,\ m}^{1}(\rho_1\theta_1,\ \rho_2\theta_2)\ g(x-i)\ s^x$$
$$+mr\rho_2\theta_2\sum_{x=0}^{\infty}\sum_{i=0}^{x}\omega_{i,\ m}^{1}(\rho_1\theta_1,\ \rho_2\theta_2)\ g(x-i)\ s^{x+m-1}$$
$$+\Lambda_2\ (r,\ \rho,\ \theta)\left[r\theta_1\sum_{x=0}^{\infty}\sum_{i=0}^{x}\ \omega_{x-i,\ m}^{r}\ (\rho_1\theta_1,\ \rho_2\theta_2)\ \omega_{i,\ m}^{r+1}\ (\theta_1,\ \theta_2)\ s^x\right.$$
$$\left.+mr\theta_2\sum_{x=0}^{\infty}\sum_{i=0}^{x}\ \omega_{x-i,\ m}^{r}\ (\rho_1\theta_1,\ \rho_2\theta_2)\ \omega_{i,\ m}^{r+1}\ (\theta_1,\ \theta_2)\ s^{x+m-1}\right], \tag{20}$$

On equating the coefficients of s^x in both sides of the expression (20), we get (17). □

3. Estimation of parameters

In this section we discuss the method of mixed moments (MMM) and method of maximum likelihood estimation (MML) for estimating the parameters of the $IGGbD(r,\ \rho_1,\ \rho_2,\ \theta_1,\ \theta_2)$.

Method of mixed moments

In MMM, the parameters are estimated by using the first four sample factorial moments and the observed relative frequency of the distribution corresponding to the observation $x = 1$. That is, the parameters are estimated by solving the equations (21), (22), (23), (24) and (25).

$$m_1 = r\delta_{21} + mr\delta_{22} + \eta^*(r\delta_{11} + mr\delta_{12}), \tag{21}$$

$$m_2 = r(r+1)(\delta_{21}^2 + m^2\delta_{22}^2 + 2m\delta_{21}\delta_{22}\eta^*(\delta_{11}^2 + m^2\delta_{12}^2) + 2m\eta^*\delta_{11}\delta_{12}$$
$$+m(m-1)r(\delta_{22} + \eta^*\delta_{12}) + 2r^2\eta^*(\delta_{11} + m\delta_{12})(\delta_{21} + m\delta_{22}), \tag{22}$$

$$
\begin{aligned}
m_3 \;=\;& (r)_3\left[\delta_{21}^3+m^3\delta_{22}^3+3m\delta_{21}\delta_{22}(\delta_{21}+m\delta_{22})+\eta^*(3m\delta_{11}^2\delta_{12}+\delta_{11}^3+\delta_{12}^3)\right]\\
&+(r)_3\eta^*3\delta_{11}\delta_{12}^2+m(m-1)(m-2)r(\delta_{22}+\eta^*\delta_{12})+3(r)_2\delta_{12}^2\delta_{22}\\
&+3(r)_2\left[m^2(m-1)(\delta_{12}^2+\delta_{22}^2)+m(m-1)(\delta_{21}\delta_{22}+\eta^*\delta_{11}\delta_{12})\right]\\
&+3m(m-1)\eta^*r^2[\delta_{11}\delta_{22}+\delta_{12}\delta_{21}+2m\delta_{12}\delta_{22}]+3r^2(r+1)\eta^*m^2\delta_{12}^2\delta_{21}\\
&+3r^2(r+1)\eta^*\left[\delta_{11}^2(\delta_{21}+m\delta_{22})+\delta_{21}^2(\delta_{11}+m\delta_{12})+m\delta_{22}^2(\delta_{11}+m^2\delta_{12})\right]\\
&+3r^2(r+1)\eta^*\left[2\delta_{12}\delta_{21}\delta_{22}+2m\delta_{11}\delta_{12}\delta_{21}+2m\delta_{11}\delta_{21}\delta_{22}+2m^3\delta_{11}\delta_{12}\delta_{22}\right], \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
m_4 \;=\;& (r)_4\left\{\delta_{21}^4+m^4\delta_{22}^4+4m\delta_{21}\delta_{22}\delta_{21}^2+m^2\delta_{22}^2+6m^2\delta_{21}^2\delta_{22}^2\right\}+(r)_4\eta^*\left\{\delta_{11}^4+m^4\delta_{12}^4\right.\\
&\left.+4m\delta_{11}\delta_{12}\delta_{11}^2+m^2\delta_{12}^2+6m^2\delta_{11}^2\delta_{12}^2\right\}+(r)_3\left\{3m^3(m-1)\eta^*\delta_{12}^3+6m(m-1)\right.\\
&\left.\left[\delta_{21}^2\delta_{22}+\eta^*\delta_{11}^2\delta_{12}\right]\right\}+12(r)_3m^2(m-1)\left[\delta_{22}^2\delta_{21}+\eta^*\delta_{12}^2\delta_{11}\right]\\
&+4m(m-1)(m-2)r^2\eta^*[\delta_{11}\delta_{22}+\delta_{12}\delta_{21}]+4(r)_2m(m-1)(m-2)\\
&[\delta_{21}\delta_{22}+\eta^*\delta_{11}\delta_{12}]+(r)_2\;m^2(m-1)(7m-11)\left[\delta_{22}^2+\eta^*\delta_{12}^2\right]\\
&+m(m-1)(m-2)(m-3)r[\delta_{22}+\eta^*\delta_{12}]+2m^2(m-1)(7m-11)r^2\eta^*\delta_{12}\delta_{22}\\
&+6m(m-1)r(r)_2\eta^*\delta_{11}[\delta_{11}\delta_{22}+2\delta_{12}\delta_{21}+2\delta_{21}\delta_{22}] \qquad (24)
\end{aligned}
$$

and

$$
\frac{p_1}{N} = \frac{(1-\rho_1\theta_1-\rho_2\theta_2)^r}{(1-\theta_1-\theta_2)^{-r}-1}r\theta_1, \qquad (25)
$$

where p_1 is the observed frequency of the distribution corresponding to the observation $x = 1$, N is the total frequency, m_1, m_2, m_3 and m_4 are the first four sample factorial moments and for any positive integer k, $(r)_k = r(r+1)\ldots(r+k-1)$ with $(r)_0 = 1$.

Method of maximum likelihood

Here we discuss the estimation of the parameters of the $IGGbD(r, \rho_1, \rho_2, \theta_1, \theta_2)$ by the MML. Let $a(x)$ be the observed frequency of x events, y be the highest value of observed x. Then the likelihood function of the sample is

$$
L = \prod_{x=0}^{y} [g(x)]^{a(x)}, \qquad (26)
$$

which implies

$$
lnL = \sum_{x=1}^{y} a(x)\; ln[g(x)]. \qquad (27)
$$

Let $\hat{r}$, $\hat{\rho_1}$, $\hat{\rho_2}$, $\hat{\theta_1}$and $\hat{\theta_2}$ denote the maximum likelihood estimators of the parameters r,

ρ_1, ρ_2, θ_1 and θ_2 of the $IGGbD(r,\ \rho_1,\ \rho_2,\ \theta_1,\ \theta_2)$ respectively. Now $\hat{r}$, $\hat{\rho_1}$, $\hat{\rho_2}$, $\hat{\theta_1}$ and $\hat{\theta_2}$ for fixed values of m are obtained by solving the likelihood equations (28), (29), (30), (31) and (32), as given below.

$\dfrac{\partial(lnL)}{\partial\theta_1} = 0$ implies

$$\sum_{x=1}^{y} a(x)\left[\frac{\phi_1^{**}(x:r,\theta_1,\theta_2,\rho_1,\rho_2)}{\phi^{**}(x:r,\theta_1,\theta_2,\rho_1,\rho_2)} - \frac{r\rho_1\theta_1}{1-\rho_1\theta_1-\rho_2\theta_2} + \frac{r\theta_1(1-\theta_1-\theta_2)^{-1}}{1-(1-\theta_1-\theta_2)^r}\right] = 0, \quad (28)$$

where $\phi^{**}(x:r,\theta_1,\theta_2,\rho_1,\rho_2) = \sum_{k=0}^{x-1} \omega_{x-k,\ m}^{r}\ (\rho_1\theta_1,\ \rho_2\theta_2)$ and

$$\phi_1^{**}(x:r,\theta_1,\theta_2,\rho_1,\rho_2) = \sum_{k=0}^{x-1}\sum_{i=0}^{\left[\frac{k}{2}\right]} \frac{\Gamma(r+k-i)\ (\rho_1\theta_1)^{k-mi}\ (\rho_2\theta_2)^i}{\Gamma(r)\Gamma(i+1)\Gamma(k-mi)} \omega_{x-k,\ m}^{r}(\theta_1,\ \theta_2)$$
$$+\sum_{k=0}^{x-1}\sum_{j=0}^{\left[\frac{x-k}{m}\right]} \frac{\Gamma(r+x-k-j)\ \theta_1^{x-k-mj}\ \theta_2^{j}}{\Gamma(r)\ \Gamma(j+1)\ \Gamma(x-k-mj)} \omega_{k,\ m}^{r}(\rho_1\theta_1,\rho_2\theta_2).$$

$\dfrac{\partial(lnL)}{\partial\theta_2} = 0$ implies

$$\sum_{x=1}^{y} a(x)\left[\frac{\phi_2^{**}(x:r,\theta_1,\theta_2,\rho_1,\rho_2)}{\phi^{**}(x:r,\theta_1,\theta_2,\rho_1,\rho_2)} - \frac{r\rho_2\theta_2}{1-\rho_1\theta_1-\rho_2\theta_2} + \frac{r\theta_2(1-\theta_1-\theta_2)^{-1}}{1-(1-\theta_1-\theta_2)^r}\right] = 0\ , \quad (29)$$

where

$$\phi_2^{**}(x:r,\theta_1,\theta_2,\rho_1,\rho_2) = \sum_{k=0}^{x-1}\sum_{i=0}^{\left[\frac{k}{m}\right]} \frac{\Gamma(r+k-i)\ (\rho_1\theta_1)^{k-mi}\ (\rho_2\theta_2)^i}{\Gamma(r)\ \Gamma(i)\ \Gamma(k-mi+1)}\ \omega_{x-k,\ m}^{r}\ (\theta_1,\ \theta_2)$$
$$+\sum_{k=0}^{x-1}\sum_{j=0}^{\left[\frac{x-k}{m}\right]} \frac{\Gamma(r+x-k-j)\ \theta_1^{x-k-mj}\ \theta_2^{j}}{\Gamma(r)\ \Gamma(j)\ \Gamma(x-k-mj+1)} \omega_{k,\ m}^{r}(\rho_1\theta_1,\rho_2\theta_2).$$

$\dfrac{\partial(lnL)}{\partial\rho_1} = 0$ implies

$$\sum_{x=1}^{y} a(x)\left[\frac{\phi_3^{**}(x:r,\theta_1,\theta_2,\rho_1,\rho_2)}{\phi^{**}(x:r,\theta_1,\theta_2,\rho_1,\rho_2)} - \frac{r\rho_1\theta_1}{1-\rho_1\theta_1-\rho_2\theta_2}\right] = 0, \quad (30)$$

where

$$\phi_3^{**}(x:r,\theta_1,\theta_2,\rho_1,\rho_2) = \sum_{k=0}^{x-1}\sum_{i=0}^{\left[\frac{k}{2}\right]} \frac{\Gamma(r+k-i)\ (\rho_1\theta_1)^{k-mi}\ (\rho_2\theta_2)^i}{\Gamma(r)\ \Gamma(i+1)\ \Gamma(k-mi)} \omega_{x-k,\ m}^{r}\ (\theta_1,\ \theta_2).$$

$\frac{\partial(lnL)}{\partial\rho_2} = 0$ implies

$$\sum_{x=1}^{y} a(x)\left[\frac{\phi_4^{**}(x:r,\theta_1,\theta_2,\rho_1,\rho_2)}{\phi^{**}(x:r,\theta_1,\theta_2,\rho_1,\rho_2)} - \frac{r\rho_2\theta_2}{1-\rho_1\theta_1-\rho_2\theta_2}\right] = 0, \tag{31}$$

$\frac{\partial(lnL)}{\partial r} = 0$ implies

$$\sum_{x=1}^{y} a(x)\left[\frac{\phi_4^{**}(x:r,\theta_1,\theta_2,\rho_1,\rho_2)}{\phi^{**}(x:r,\theta_1,\theta_2,\rho_1,\rho_2)} + \frac{ln(1-\theta_1-\theta_2)}{1-(1-\theta_1-\theta_2)^r} + ln(1-\rho_1\theta_1-\rho_2\theta_2)\right] = 0, \tag{32}$$

where

$$\phi_4^{**}(x:r,\theta_1,\theta_2,\rho_1,\rho_2) = \sum_{k=0}^{x-1}\sum_{i=0}^{\left[\frac{k}{m}\right]} \frac{\Gamma(r+k-i)\ (\rho_1\theta_1)^{k-mi}\ (\rho_2\theta_2)^i}{\Gamma(r)\ \Gamma(i)\ \Gamma(k-mi+1)}\ \omega_{x-k,\ m}^{r}\,(\theta_1,\ \theta_2)$$

and $\phi^{**}(x:r,\theta_1,\theta_2,\rho_1,\rho_2)$ is as defined in (28). Note that these likelihood equations do not always have unique solutions. The maximum of the likelihood function is attained at the border of domain of the parameters. We have obtained the second order partial derivatives of lnL with respect to the parameters r, ρ_1, ρ_2, θ_1 and θ_2 by using MATHEMATICA software and it is observed that these equations gives negative values, for all $r > 0$, $\rho_1 > 0$, $\rho_2 > 0$, $\theta_1 > 0$ and $\theta_2 > 0$. Thus the model $IGGbD(r,\ \rho_1,\ \rho_2,\ \theta_1,\ \theta_2)$ is log concave and under this parametric restriction the maximum likelihood estimates are unique. For solving the likelihood equations we can use the mathematical software MATHEMATICA.

4. Testing of hypothesis

Here we discuss the generalized likelihood ratio test (GLRT) procedure for testing the significance of the parameter ρ_1 and ρ_2 of the $IGGbD(r,\ \rho_1,\ \rho_2,\ \theta_1,\ \theta_2)$. We consider the following tests:

Test1 : $H_{01}:\rho_1=\rho_{10}$ vs $H_{11}:\rho_1>\rho_{10}$,
Test2 : $H_{01}:\rho_2=\rho_{20}$ vs $H_{11}:\rho_2>\rho_{20}$,
Test3 : $H_{01}:\rho_1=\rho_{10},\rho_2=\rho_{20}$ vs $H_{11}:\rho_1>\rho_{10},\rho_2>\rho_{20}$,

where ρ_{10} and ρ_{20} are fixed positive real number very close to zero . Hence the test statistic is

$-2ln\lambda = 2\left[ln(L(\underline{\hat{\theta}},x)) - ln(L(\underline{\hat{\theta}}^*,x))\right]$,

where $\underline{\hat{\theta}}$ is the maximum likelihood estimator of the parameter set $\theta=(r,\rho_1,\rho_2,\theta_1,\theta_2)$ with no restriction and $\hat{\theta}^*$ is the maximum likelihood estimator of $\underline{\theta}$ under H_0. Clearly, $-2ln\lambda$ is asymptotically distributed as χ^2 distribution with k degrees of freedom, where k is the number of restrictions under H_0.

4.1. Numerical illustration

In order to illustrate the usefulness of the model IGGbD, we have considered two real life data sets- Data Set- 1 and Data Set-2 . The model IGGbD is fitted to that data sets by MMM and MML. We have also checked the goodness of fit and compared the model IGGbD with some existing competing intervened type models such as IGD, INBD along with ZTNBD. The Data Set-1 is the distribution of number of moth species represented by n individuals in a sample from a heavily logged rain forest. For details of Data set- 1 see [2]. The Data set-2 is the distribution of 1534 biologists according to the number of research papers to their credit in the Review of Applied Entomology taken from [6] . The numerical results obtained are summarised in Tables 3 to 6. From these tables it is clear that the IGGbD with $m = 4$ give better fit to the Data Set 1 while IGGbD with $m = 3$ give better fit to the Data Set 2 compared to IGD, ZTNBD and INBD. In order to check the significance of the intervention parameters of the IGGbD for both the data sets, we have computed the value of the test statistic for GLRT and are given in Table 1 and 2. From these tables it is clear that the intervention parameters are significant in both cases.

Table 1.: Computed values of the GLRT statistic for the data set given in Table 3

	$log(L(\hat{\theta}, x))$	$log(L(\hat{\theta}^*, x))$	Test statistic value	df	P-value
Test1	-253.90	-260.78	13.65	1	0.0002
Test2	-253.90	300.28	92.77	1	<0.00001
Test3	-253.90	263.55	19.31	2	6.4E-05

Table 2.: Computed values of the GLRT statistic for the data set given in Table 5

	$log(L(\hat{\theta}, x))$	$log(L(\hat{\theta}^*, x))$	Test statistic value	df	P-value
Test1	-1524.92	-1527.26	4.68	1	0.0305
Test2	-1524.92	-1527.82	5.80	1	0.0160
Test3	-1524.92	-1529.18	8.52	2	0.0141

Table 3.: Observed distribution of number of moth species represented by n individuals in a sample from a heavily logged rain forest, reported by [2] and expected frequencies by MMM.

		Expected frequencies						
					IGGbD for different values of m			
x	f	IGD	ZTNBD	INBD	m=2	m=3	m=4	m=5
1	94	66.65	49.83	82.80	81.39	83.19	86.85	81.80
2	25	40.73	50.11	32.76	29.01	31.25	27.35	27.94
3	13	23.72	34.31	17.28	14.62	18.01	11.77	11.77
4	7	13.78	17.01	10.26	9.22	10.59	9.10	5.88
5	6	8.00	7.70	6.49	6.45	6.63	7.37	6.73
6	2	4.65	2.80	4.28	3.62	4.34	5.27	6.49
7	2	2.70	0.89	2.90	2.81	2.92	3.71	4.91
8	5	1.56	0.25	2.01	2.01	2.01	2.70	3.79
9	10	2.21	1.10	5.22	14.87	5.06	9.88	14.69
Total	164	164	164	164	164	164	164	164
df		3	3	3	1	1	1	1
Estimated value of parameters		$\hat{\rho} = 19.1707$ $\hat{\theta} = 0.0303$	$r = 45.335$ $\hat{\theta} = 0.0434$	$\hat{r} = 3.03 * 10^{-7}$ $\hat{\rho} = 0.8691$ $\hat{\theta} = 0.7914$	$\hat{r} = 0.2942$ $\hat{\rho_1} = 7.6431$ $\hat{\rho_2} = 8.2445$ $\hat{\theta_1} = 0.1073$ $\hat{\theta_2} = 0.0049$	$\hat{r} = 4.42 * 10^{-6}$ $\hat{\rho_1} = 0.4328$ $\hat{\rho_2} = 14.069$ $\hat{\theta_1} = 0.7513$ $\hat{\theta_2} = 0.0213$	$\hat{r} = 0.3625$ $\hat{\rho_1} = 6.5726$ $\hat{\rho_2} = 24.827$ $\hat{\theta_1} = 0.1028$ $\hat{\theta_2} = 0.0036$	$\hat{r} = 0.3299$ $\hat{\rho_1} = 3.8923$ $\hat{\rho_2} = 14.87$ $\hat{\theta_1} = 0.1753$ $\hat{\theta_2} = 0.0075$
Chi- square value		31.563	109.904	9.867	4.886	10.156	1.994	6.552
P - value		$1.27E-14$	$6.47E-07$	0.007	0.027	0.00014	0.158	0.01
AIC		545.096	610.036	521.786	526.40	524.59	520.848	530.976
BIC		551.296	616.236	524.986	526.60	524.792	521.048	531.176
AICc		541.17	606.11	514.93	516.77	514.967	511.223	521.351

Table 4.: Observed distribution of number of moth species represented by n individuals in a sample from a heavily logged rain forest, reported by [2] and expected frequencies by MML.

		Expected frequencies						
					IGGbD for different values of m			
x	f	IGD	ZTNBD	INBD	m=2	m=3	m=4	m=5
1	94	65.34	83.74	92.64	82.80	83.20	92.42	91.46
2	25	41.96	32.87	27.62	32.76	31.26	27.61	31.18
3	13	24.29	17.20	13.86	17.28	18.00	12.55	13.26
4	7	13.89	10.13	8.54	10.26	10.58	8.38	6.61
5	6	7.93	6.36	5.71	6.49	6.63	6.21	5.26
6	2	5.07	5.02	3.99	5.07	5.12	6.31	5.01
7	2	2.58	2.80	3.87	2.90	2.92	3.00	3.32
8	5	1.49	1.92	2.10	2.01	2.01	2.13	2.34
9	10	1.45	3.96	5.67	4.43	4.28	5.39	5.56
Total	164	164	164	164	164	164	164	164
df		4	4	2	1	1	1	1
Estimated value of parameters		$\hat{\rho} = 8.0201$ $\hat{\theta} = 0.0712$	$r = 0.000099$ $\hat{\theta} = 0.7853$	$\hat{r} = 0.3149$ $\hat{\rho} = 12.6398$ $\hat{\theta} = 0.0643$	$\hat{r} = 1.16E-06$ $\hat{\rho_1} = 0.3763$ $\hat{\rho_2} = 8.2445$ $\hat{\theta_1} = 0.7914$ $\hat{\theta_2} = 1.46E-06$	$\hat{r} = 1.009E-05$ $\hat{\rho_1} = 0.3589$ $\hat{\rho_2} = 14.069$ $\hat{\theta_1} = 0.7514$ $\hat{\theta_2} = 0.0212$	$\hat{r} = 0.3546$ $\hat{\rho_1} = 7.3499$ $\hat{\rho_2} = 29.6977$ $\hat{\theta_1} = 0.0943$ $\hat{\theta_2} = 0.0018$	$\hat{r} = 0.3735$ $\hat{\rho_1} = 4.7669$ $\hat{\rho_2} = 29.0031$ $\hat{\theta_1} = 0.1382$ $\hat{\theta_2} = 0.0019$
Chi- square value		35.24	8.511	9.237	8.579	8.324	1.697	4.738
P - value		$4.14E-07$	0.075	0.026	0.0034	0.0039	0.193	0.03
lnL		-272.56	-258.73	-257.97	-257.39	-257.29	-253.90	-255.01
AIC		549.12	521.46	521.94	524.78	524.58	517.8	520
BIC		555.32	527.66	526.14	524.98	524.78	518	520
AICc		545.19	517.53	516.08	515.96	514.95	508.17	510

Table 5.: Observed distribution of 1534 biologists according to the number of research papers to their credit in the Review of Applied Entomology taken from [6] and expected frequencies by MML.

		Expected frequencies					
					IGGbD for different values of m		
x	f	IGD	ZTNBD	INBD	m=2	m=3	m=4
1	1062	991.53	1052.00	1035,00	1056.19	1063.47	1051.20
2	263	356.37	296.19	296.12	270.38	262.77	289.13
3	120	122.34	107.81	112	125.27	119.46	106.03
4	50	41.91	43.68	48.20	43.55	50.06	47.18
5	22	14.35	18.78	20.22	21.15	20.97	21.14
6	7	4.91	8.38	9.01	10.13	8.44	9.86
7	6	1.68	3.84	5.13	4.54	4.37	4.73
8	2	0.57	1.79	2.60	2.16	2.04	2.32
9	0	0.19	0.85	1.33	1.09	1.96	1.15
10+	2	0.15	0.70	1.39	0.54	0.46	1.26
Total	1534	1534	1534	1534	1534	1534	1534
df		3	4	4	1	1	1
Estimated value		$\hat{\rho} = 20.26$	$r = 0.0650$	$\hat{r} = 0.0014$	$\hat{r} = 0.3583$	$\hat{r} = 0.4806$	$\hat{r} = 1.085E-06$
of parameters		$\hat{\theta} = 0.0169$	$\hat{\theta} = 0.5288$	$\hat{\rho} = 0.981$	$\hat{\rho_1} 1.3814$	$\hat{\rho_1} = 7.2221$	$\hat{\rho_1} = 0.6326$
				$\hat{\theta} = 0.569$	$\hat{\rho_2} = 15.4635$	$\hat{\rho_2} = 9.9035$	$\hat{\rho_2} = 28.482$
					$\hat{\theta_1} = 0.1527$	$\hat{\theta_1} = 0.0587$	$\hat{\theta_1} = 0.5501$
					$\hat{\theta_2} = 0.0117$	$\hat{\theta_2} = 0.0022$	$\hat{\theta_2} = 0.0018$
Chi- square value		67.94	8.172	6.223	3.689	0.411	5.35
P - value		$6.15E-14$	0.085	0.1012	0.055	0.521	0.021
lnL		-1541.87	-1529.76	-1528.02	-1525.48	-1524.92	-1527.95
AIC		3088	3064	3062	3061	3060	3066
BIC		3098	3074	3071	3066	3065	3071
AICc		3084	3060	3056	3051	3050	3056

Table 6.: Observed distribution of 1534 biologists according to the number of research papers to their credit in the Review of Applied Entomology taken from [6] and expected frequencies by MMM.

		Expected frequencies					
					IGGbD for different values of m		
x	f	IGD	ZTNBD	INBD	m=2	m=3	m=4
1	1062	978.28	970.40	1077.10	1042.66	1066.36	1051.88
2	263	369.29	356.11	285.06	211.46	266.26	289.13
3	120	124.51	131.01	101.07	132.33	107.65	106.03
4	50	41.38	48.26	48.10	78.62	53.56	47.18
5	22	13.72	17.79	17.16	31.18	22.55	21.14
6	7	4.55	6.56	6.01	15.32	7.70	9.86
7	6	1.50	2.42	3.12	10.06	4.49	4.73
8	2	0.50	0.33	2.43	6.51	2.06	2.32
9	0	0.16	0.12	1.17	3.98	1.94	1.15
10+	2	0.11	1.00	0.78	1.88	1.43	0.58
Total	1534	1534	1534	1534	1534	1534	1534
df		3	4	3	2	1	1
Estimated value		$\hat{\rho} = 7.2242$	$r = 0.9847$	$\hat{r} = 0.0014$	$\hat{r} = 0.2695$	$\hat{r} = 0.6368$	$\hat{r} = 3.66E-06$
of parameters		$\hat{\theta} = 0.0459$	$\hat{\theta} = 0.3698$	$\hat{\rho} = 0.95$	$\hat{\rho_1} 1.7406$	$\hat{\rho_1} = 9.0649$	$\hat{\rho_1} = 0.7976$
				$\hat{\theta} = 0.529$	$\hat{\rho_2} = 9.3948$	$\hat{\rho_2} = 26.973$	$\hat{\rho_2} 33.35$
					$\hat{\theta_1} = 0.1839$	$\hat{\theta_1} = 0.0379$	$\hat{\theta_1} = 0.5501$
					$\hat{\theta_2} = 0.0153$	$\hat{\theta_2} = 0.0012$	$\hat{\theta_2} = 0.0018$
Chi- square value		58.99	38.876	10.11	39.60	1.785	5.35
P - value		$1.95E-11$	$1.58E-07$	0.0175	$2.51E-09$	0.181	0.021
lnL		-1547.98	-1548.29	-1528.78	-1528.08	-1525.73	-1527.25
AIC		3100	3101	3064	3066	3061	3065
BIC		3111	3111	3072	3071	3066	3069
AICc		3096	3097	3058	3056	3051	3055

5. Simulation

For examining the performance of the estimators, we have simulated data sets from the IGGbD with $m = 3$ and $m = 4$ for the following parameter set. For $m = 3$, we have used the parameter set i) $r = 1.802$, $\rho_1 = 1.3584$, $\rho_2 = 1.37$, $\theta_1 = 0.1031$, $\theta_2 = 0.1216$ (Over-dispersion) and ii) $r = 0.802$, $\rho_1 = 4.3584$, $\rho_2 = 1.58$, $\theta_1 = 0.02$, $\theta_2 = 0.0216$ (Under-dispersion). For $m = 4$, the parameter set we used is i) $r = 1.18$, $\rho_1 = 0.768$, $\rho_2 = 2.01$, $\theta_1 = 0.57$, $\theta_2 = 0.0457$ (Over-dispersion) and ii) $r = 0.26$, $\rho_1 = 2.125$, $\rho_2 = 0.495$, $\theta_1 = 0.02$ and $\theta_2 = 0.0234$ (Under-dispersion). We have computed the bias and standard error in each case by estimating the parameters by MMM and MML, which is presented in Table 7 and 8. From these tables it is seen that as the sample size increases, the values absolute bias and standard errors are decreases.

Table 7.: Bias and standard errors (within brackets) of MMM estimators of the parameters of the IGGbD using simulated data sets.

	parameters	Over-dispersion Sample size 500	Over-dispersion Sample size 1000	Under-dispersion Sample size 500	Under-dispersion Sample size 1000
m=3	r	-0.9721	-0.8752	0.9870	0.9615
		(0.7016)	(0.4528)	(0.2086)	(0.1926)
	ρ_1	0.8546	0.6781	-0.8812	-0.8449
		(0.1526)	(0.1732)	(0.0698)	(0.0426)
	ρ_2	0.7234	0.6634	-0.6457	-0.6139
		(0.0768)	(0.0456)	(0.2815)	(0.2189)
	θ_1	0.1213	0.1168	0.1779	0.1695
		(0.0012)	(0.0001)	(0.0014)	(0.0009)
	θ_2	-0.0902	-0.0834	-0.0223	-0.0129
		(0.0125)	(0.0047)	(0.0176)	(0.0058)
m=4	r	-0.0156	-0.0123	-0.0867	-0.0853
		(0.0057)	(0.0012)	(0.0014)	(0.0002)
	ρ_1	0.5819	0.5621	0.6678	0.6247
		(0.0024)	(0.0003)	(0.0019)	(0.0001)
	ρ_2	1.0123	0.9998	-0.9234	-0.9128
		(0.0345)	(0.0312)	(0.0276)	(0.0079)
	θ_1	0.3576	0.2914	0.3019	0.2721
		(0.0789)	(0.0656)	(0.0728)	(0.0432)
	θ_2	0.0126	0.0107	0.0175	0.0142
		(0.0125)	(0.0017)	(0.1008)	(0.0723)

Table 8.: Bias and standard errors (with in brackets) of MML estimators of the parameters of the IGGbD using simulated data sets.

	parameters	Over-dispersion Sample size 500	Over-dispersion Sample size 1000	Under-dispersion Sample size 500	Under-dispersion Sample size 1000
m=3	r	-0.7529 (0.6085)	-0.5632 (0.1214)	1.0123 (0.1073)	0.9934 (0.0934)
	ρ_1	0.7732 (0.2565)	0.4801 (0.1725)	-0.8903 (0.0767)	-0.8340 (0.0034)
	ρ_2	0.5219 (0.1808)	0.4602 (0.1214)	-0.3457 (0.1938)	-0.2139 (0.1761)
	θ_1	0.0315 (0.0091)	0.0182 (0.0071)	0.0973 (0.0026)	0.0694 (0.0021)
	θ_2	-0.0101 (0.0066)	-0.0026 (0.0017)	-0.0219 (0.0129)	-0.0116 (0.0098)
m=4	r	-0.0207 (0.0167)	-0.0172 (0.0112)	-0.0934 (0.0034)	-0.0896 (0.0022)
	ρ_1	0.4895 (0.0087)	0.3621 (0.0007)	0.7648 (0.0021)	0.6238 (0.0002)
	ρ_2	1.1810 (0.0297)	0.9162 (0.0081)	-0.8102 (0.0258)	-0.7134 (0.0062)
	θ_1	0.2145 (0.0285)	0.1934 (0.0179)	0.2018 (0.0745)	0.1756 (0.0196)
	θ_2	0.0022 (0.0075)	0.0019 (0.0006)	0.0189 (0.1023)	0.0024 (0.0945)

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